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On the convergence of finite element solutions to the interface problem for the Stokes system☆

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Abstract

The Stokes system with a discontinuous coefficient (Stokes interface problem) and its finite element approximations are considered. We firstly show a general error estimate. To derive explicit convergence rates, we introduce some appropriate assumptions on the regularity of exact solutions and on a geometric condition for the triangulation. We mainly deal with the MINI element approximation and then consider P1-iso-P2/P1 element approximation. Results are expected to give an instructive remark in numerical analysis for two-phase flow problems.

Keywords: Stokes equations; Interface problem; Finite element method

1. Introduction

In numerical approximations of multi-phase flow problems of viscous incompressible fluids, after a discretization of the time variable or a linearization of the nonlinear term, we often meet the Stokes system where the (kinematic) coefficient of viscosity $\nu = \nu(x)$ is a piecewise constant function. The discontinuity of ν leads to the interface. Hence we call it a *Stokes interface problem*. The purpose of this paper is to give convergence analysis of finite element approximations for such problem. There are sophisticated studies devoted to finite element methods for scalar elliptic interface problems; for example, we can refer to [2,4,6,8,22,23]. However, it seems that little is known for the Stokes case. In the present paper, we concentrate our attention to simple finite element methods and mainly deal with the MINI element approximation. The interface is approximated by a piecewise linear curve. We firstly establish a convergence theorem (Theorem 3) under a general triangulation without any regularity assumptions of solutions. To derive the rate of convergence, however, we need some assumptions on the regularity of solutions and also on the triangulation. First, since regularity theory for a Stokes interface problem is not complete at present, we introduce a somewhat artificial assumption in terms of fractional order Sobolev spaces as

$$\text{the velocity } u \in H^{1+\varepsilon}, \quad \text{the pressure } p \in H^{\varepsilon} \text{ for some } \varepsilon \in (0, 1]$$

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on each subdomains in which ν is constant. It is an analogy of a Poisson interface problem (see Appendix A). When $\varepsilon \in (0, \frac{1}{2})$, such assumption implies the global regularity, and we immediately obtain an error estimate of the order h^ε (Theorem 7), $h > 0$ being the size of the spatial discretization. On the other hand, in the case of $\varepsilon \in [\frac{1}{2}, 1]$, the usual argument of interpolation error estimates does not work. To overcome this issue, we follow the method of [22,23] and introduce a geometrical condition on the triangulation. Under these assumptions, we can obtain an error estimate of the order h^ε (Theorem 9). Finally, an extension to the P1-iso-P2/P1 element is also discussed (Theorem 12). Theorems 3 and 7 could be extended to all popular conforming elements satisfying the uniform inf–sup condition (Remark 4). However, since the interface is approximated by a piecewise linear curve, we only obtain the first order convergence (with respect to the H^1 norm for the velocity and the L^2 norm for the pressure), even if higher order elements are employed (Remark 5). We should note that error estimates in Theorems 7 and 9 are of optimal order under our regularity assumptions of a solution. Moreover, Theorems 3 and 7 are established under a general triangulation and that they are new even in the literature of elliptic interface problems.

As was stated above, a Stokes interface problem is closely related with multi-phase fluid flow problems, and their numerical solutions are now developing subjects [13,17]. It seems that main efforts to these problems are concentrated on a reproduction of the moving interface between different fluids. In contrast, there is a room for further study on the effect of discontinuities of density and viscosity on approximate solutions. Therefore, results of this paper are expected to give an instructive remark for those practical applications. See, for example, Remark 13.

Our presented scheme and error estimates are not *robust*, which means that the error estimate becomes worse when the ratio $\alpha = \max \nu(x)/ \min \nu(x)$ is large enough (Remark 6). Robust error estimates play an important role to design effective numerical schemes, as is discussed in [16]. Therefore, our result may be unsatisfactory from the view point of effective computations. However, our motivation of this work is to reveal a relationship between a general nature of convergence of some finite element solutions and the discontinuity of a coefficient.

After this work was completed, the authors learned of the work of Girault et al. [10], which treats the similar problem as the present paper. They take into account the effect of the surface tension, whereas we neglect it for the sake of simplicity. However, the final error estimate in [10] seems to be not optimal.

This paper is composed of six sections. In Section 2, we formulate a Stokes interface problem, and then Section 3 describes a finite element scheme by the MINI element approximation. In Section 4, a general error estimate (Theorem 3) is stated. Section 5 is devoted to error estimates with explicit convergence rates and Theorems 7 and 9 are described there. An extension to P1-iso-P2/P1 element is discussed in Section 6. We conclude this paper by giving a brief review of the regularity results for Stokes and Poisson interface problems in Appendix A.

Notation

Throughout this paper, we follow the notation of [15]. Namely, we use standard Lebesgue and Sobolev spaces $L^2(\mathcal{O})$, $H^1(\mathcal{O})$ and $H_0^1(\mathcal{O})$, where $\mathcal{O}$ denotes a domain in $\mathbb{R}^2$. Product spaces are written as, for example, $L^2(\mathcal{O})^2 = L^2(\mathcal{O}) \times L^2(\mathcal{O})$, and we use the same symbol to denote the norm of product spaces; $\|\cdot\|_{L^2(\mathcal{O})} = \|\cdot\|_{L^2(\mathcal{O})^2}$. Also, for tensor functions of $L^2(\mathcal{O})^4$, we write as $\|\cdot\|_{L^2(\mathcal{O})} = \|\cdot\|_{L^2(\mathcal{O})^4}$. Let m be a nonnegative integer, and let $\varepsilon \in (0, 1)$. In [15], the fractional order Sobolev space $H^{m+\varepsilon}(\mathcal{O})$ is defined as the interpolation space between $H^m(\mathcal{O})$ and $H^{m+1}(\mathcal{O})$. ($H^0(\mathcal{O})$ is understood as $L^2(\mathcal{O})$.) However, it is worth-while noting that $H^{m+\varepsilon}(\mathcal{O})$ is equipped with the equivalent norm

$$\|v\|_{H^{m+\varepsilon}(\mathcal{O})} = \left[\|v\|^2_{H^m(\mathcal{O})} + \sum_{|\alpha|=m} \iint_{\mathcal{O}\times\mathcal{O}} \frac{|\partial_x^\alpha v(x) - \partial_y^\alpha v(y)|^2}{|x-y|^{2+2\varepsilon}} \, \mathrm{d}x \, \mathrm{d}y \right]^{1/2},$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index with $|\alpha| = \alpha_1 + \alpha_2$ and

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}, \quad \partial_y^\alpha = \frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1} \partial y_2^{\alpha_2}}.$$

We also use a weighted Sobolev space $H_{00}^{1/2}(S)$, where S is a curve in $\mathbb{R}^2$. For the precise definition, see [15]. Generic positive constants depending on parameters $\gamma_1, \gamma_2, \ldots$ are denoted by $C = C(\gamma_1, \gamma_2, \ldots)$. Moreover, in general, H^* denotes the dual space of a Hilbert space H.

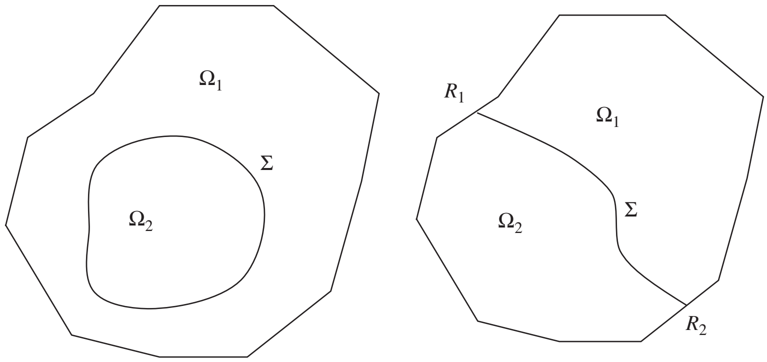

Fig. 1. Examples of Ω. $\partial\Omega \cap \overline{\Sigma} = \emptyset$ (left); $\partial\Omega \cap \overline{\Sigma} \neq \emptyset$ (right).

2. Stokes interface problem

Let Ω be a polygonal domain in $\mathbb{R}^2$. We suppose that Ω is divided into two disjoint subdomains Ω_1 and Ω_2 by a C^2 curve $\Sigma = \partial\Omega_1 \cap \partial\Omega_2$. The interface Σ may be closed, i.e. $\partial\Omega \cap \overline{\Sigma} = \emptyset$. In the case where $\partial\Omega \cap \overline{\Sigma} \neq \emptyset$, we assume that $\partial\Omega$ and Σ meet at R_1 and R_2 transversally. As a consequence, $\partial\Omega_1$ and $\partial\Omega_2$ are Lipschitz curves. Further we suppose that $\overline{\Sigma}$ and $\partial\Omega$ has no intersection points except for R_1 and R_2. For example, see Fig. 1. The viscosity coefficient is given by a discontinuous function

$$v(x) = \begin{cases} v_1 & (x \in \Omega_1) \\ v_2 & (x \in \Omega_2), \end{cases}$$

where v_1 and v_2 are positive constants such that $v_1 > v_2$. Set

$$V = H_0^1(\Omega)^2, \quad \|\cdot\|_V = \|\cdot\|_{H^1(\Omega)},$$

$$Q = \left\{ q \in L^2(\Omega) \,\middle|\, \int_\Omega q \,\mathrm{d}x = 0 \right\}, \quad \|\cdot\|_Q = \|\cdot\|_{L^2(\Omega)}.$$

We introduce a bilinear form on $V \times V$:

$$a(u, v) = \int_\Omega 2v(x) D_{ij}(u) D_{ij}(v) \,\mathrm{d}x \quad (u, v \in V),$$

where $D_{ij}(u) = (\frac{1}{2})(\partial u^i/\partial x_j + \partial u^j/\partial x_i)$ for $u = (u^1, u^2) \in V$ and the summation convection is employed. We also use

$$b(v, p) = -\int_\Omega p \operatorname{div} v \,\mathrm{d}x \quad (v \in V, p \in Q),$$

$$(u, v) = \int_\Omega u \cdot v \,\mathrm{d}x \quad (u, v \in L^2(\Omega)^2 \text{ or } u, v \in L^2(\Omega)).$$

Let $f \in L^2(\Omega)^2$ be given. Then we consider a Stokes interface problem: find $(u, p) \in V \times Q$ satisfying

$$\begin{cases} a(u, v) + b(v, p) = (f, v) & (\forall v \in V), \\ b(u, q) = 0 & (\forall q \in Q). \end{cases} \tag{1}$$

We recall that there are $\delta_i = \delta_i(\Omega)$, $i = 1, \ldots, 4$, such that

$$|a(u, v)| \leqslant v_1 \delta_1 \|u\|_V \|v\|_V \quad (u, v \in V), \tag{2}$$

$$v_2 \delta_2 \|u\|_V^2 \leqslant a(u, u) \quad (u \in V), \tag{3}$$

$$|b(v, p)| \leqslant \delta_3 \|v\|_V \|p\|_Q \quad (v \in V, p \in Q), \tag{4}$$

$$\sup_{v \in V} \frac{b(v, q)}{\|v\|_V} \geqslant \delta_4 \|q\|_Q \quad (q \in Q). \tag{5}$$

Especially, Inequality (3) is a consequence of Korn's inequality. As a result, according to the well-known saddle point theory (cf. [5, Theorem 1.1], [9, Theorem 5.6], [11, Corollary 4.1]), Problem (1) admits a unique solution $(u, p) \in V \times Q$.

Moreover it is easy to verify (cf. [19]) that the solution $(u, p) \in V \times Q$ of (1) satisfies

$$-v_k \Delta u_k + \nabla p_k = f|_{\Omega_k} \quad \text{in } [H_0^1(\Omega_k)^2]^* \ \ (k = 1, 2),$$

where $u_k = u|_{\Omega_k}$ and $p_k = p|_{\Omega_k}$. Furthermore, on Σ, we have

$$u_1 = u_2 \quad \text{in } H^{1/2}(\Sigma)^2, \tag{6}$$

$$\sigma_1 n = \sigma_2 n \quad \text{in } W, \tag{7}$$

where $\sigma_k = [-p_k \delta_{ij} + 2v_k D_{ij}(u_k)]$ is the stress tensor associated with (u_k, p_k), δ_{ij} is Kronecker's delta, $n = (n_1, n_2)$ is the unit vector to Σ outgoing from Ω_1, and

$$W = \begin{cases} [H^{1/2}(\Sigma)^2]^* & (\text{if } \partial\Omega \cap \overline{\Sigma} = \emptyset), \\ [H_{00}^{1/2}(\Sigma)^2]^* & (\text{if } \partial\Omega \cap \overline{\Sigma} \neq \emptyset). \end{cases}$$

In particular, (6) and (7) imply the continuity of the flow velocity and the stress vector across Σ, respectively. Therefore, we might say that (1) is a simple model of a stationary two-phase problem of viscous incompressible fluids without surface tension on the interface.

3. Finite element scheme. MINI element approximation

Let $\{\mathscr{T}_h\}_h$ be a family of quasi-uniform triangulations of Ω; there are $\kappa_1 > 0$ and $\kappa_2 > 0$ satisfying

$$\kappa_1 h \leqslant d_T \leqslant \kappa_2 \rho_T \quad (\forall T \in \mathscr{T}_h \in \{\mathscr{T}_h\}), \tag{8}$$

where d_T denotes the diameter of K, ρ_T the diameter of the inscribed ball of T, and $h = \max\{d_K |\ K \in \mathscr{T}_h\}$. Each triangle $T \in \mathscr{T}_h$ is assumed to be a closed set. For any $T \in \mathscr{T}_h$, let $\mathscr{P}_n(T)$ be the set of all polynomials defined on T of degree $\leqslant n$, and let $\mathscr{B}(T) = [\mathscr{P}_1(T) \oplus \text{span}\{\lambda_1\lambda_2\lambda_3\}]^2$, where λ_i are the barycentric coordinates of T. Put

$$V_h = \{v_h \in C(\overline{\Omega})^2 \cap V \mid v_h|_T \in \mathscr{B}(T) \ (\forall T \in \mathscr{T}_h)\},$$
$$Q_h = \{q_h \in C(\overline{\Omega}) \cap Q \mid q_h|_T \in \mathscr{P}_1(T) \ (\forall T \in \mathscr{T}_h)\}.$$

It is well-known (cf. [11]) that a pair of V_h and Q_h satisfies the uniform inf–sup condition

$$\sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V} \geqslant \delta_5 \|q_h\|_Q \quad (q_h \in Q_h), \tag{9}$$

where $\delta_5 = \delta_5(\Omega) > 0$ is independent of h. We will not deal with the standard finite element approximation of (1) and take into account approximations of $v(x)$. Let τ_h be the set of interfacial elements defined by

$$\tau_h = \{T \in \mathscr{T}_h | \, (\text{Int}\, T) \cap \Sigma \neq \emptyset\},$$

where $\text{Int}\, T$ denotes the interior region of T. Since Σ is of class C^2, there is $h_0 > 0$ such that, for any $T \in \tau_h$ and $h \in (0, h_0)$, ∂T and Σ connect by only two points P_T^1 and P_T^2 (Fig. 2). For each $T \in \tau_h$, we define

$$l_T = \text{the line segment } \overline{P_T^1 P_T^2} \subset T;$$

$$\omega_T = \text{the open region enclosed by } \Sigma \text{ and } l_T.$$

By re-choosing h_0 if necessary, we may assume that there is $\kappa_3 = \kappa_3(\Sigma, h_0) > 0$ satisfying

$$\text{dist}\,(x, l_T) \leqslant \kappa_3 h^2 \quad (x \in \omega_T, \ T \in \tau_h) \tag{10}$$

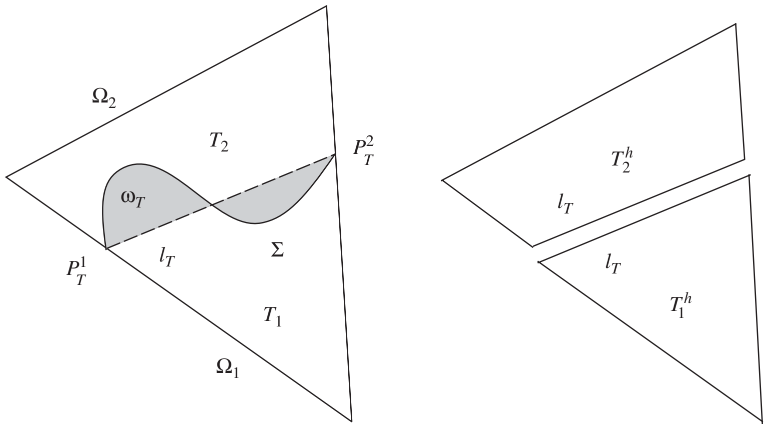

Fig. 2. P_1^T, P_2^T, l_T, ω_T, T_1, T_2, T_1^h and T_2^h for a typical element $T \in \tau_h$.

for $h \in (0, h_0)$. In the sequel, we always suppose $h \in (0, h_0)$ unless otherwise stated. It is easy to see that $\bigcup_{T \in \tau_h} l_T$ is a piecewise linear approximation of Σ. We further set, for each $T \in \tau_h$,

$$T_k = \overline{T \cap \Omega_k} \quad (k = 1, 2),$$

$$T_k^h = \overline{(T_k \backslash \omega_T) \cup (T_m \cap \omega_T)} \quad (k, m = 1, 2;\ k \neq m).$$

Then we introduce an approximation of $v(x)$ by setting

$$v_h(x) = \begin{cases} v(x) & (x \in \operatorname{Int} T,\ T \in \mathscr{T}_h \backslash \tau_h), \\ v_k & (x \in \operatorname{Int} T_k^h,\ k = 1, 2,\ T \in \tau_h) \end{cases}$$

and

$$a_h(u_h, v_h) = \int_\Omega 2v_h(x) D_{ij}(u_h) D_{ij}(v_h)\, \mathrm{d}x \quad (u_h, v_h \in V_h).$$

Now we can state our finite element approximation of (1): find $(u_h, p_h) \in V_h \times Q_h$ satisfying

$$\begin{cases} a_h(u_h, v_h) + b(v_h, p_h) = (f, v_h) & (\forall v_h \in V_h), \\ b(u_h, q_h) = 0 & (\forall q_h \in Q_h). \end{cases} \tag{11}$$

Since $a_h(\cdot, \cdot)$ satisfies

$$|a_h(u_h, v_h)| \leqslant v_1 \delta_1 \|u\|_V \|v\|_V \quad (u_h, v_h \in V_h), \tag{12}$$

$$v_2 \delta_2 \|u_h\|_V^2 \leqslant a_h(u_h, u_h) \quad (u_h \in V_h) \tag{13}$$

with the same constants as in (2) and (3), the approximate problem (11) admits a unique solution $(u_h, p_h) \in V_h \times Q_h$.

4. Error estimate

In this section, we state a convergence theorem (Theorem 3) under a general quasi-uniform triangulation without any regularity assumptions of solutions. We begin with the following lemmas.

Lemma 1. *There is a constant $C = C(\kappa_1, \kappa_2, \kappa_3, l) > 0$ such that*

$$\|\chi\|_{L^2(\omega_T)} \leqslant C\sqrt{h} \|\chi\|_{L^2(T)} \tag{14}$$

for any $T \in \mathscr{T}_h$ and any polynomial χ of the degree $l \geqslant 0$ defined on T.

We skip the proof, because it can be done in the essentially same way as that of [21, Lemma 2].

Lemma 2. *There is a constant* $c_0 = c_0(\kappa_1, \kappa_2, \kappa_3) > 0$ *such that*

$$\sup_{v_h \in V_h} \frac{|a_h(w_h, v_h) - a(w_h, v_h)|}{\|v_h\|_V} \leqslant c_0(v_1 - v_2)h\|w_h\|_V \tag{15}$$

for any $w_h \in V_h$.

Proof. By virtue of (14), for any $T \in \mathscr{T}_h$ and $v_h \in V_h$,

$$\|\nabla v_h\|^2_{L^2(\omega_T)} \leqslant Ch\|\nabla v_h\|^2_{L^2(T)},$$

where $C = C(\kappa_1, \kappa_2, \kappa_3)$. Hence, we have, for any $w_h, v_h \in V_h$,

$$\begin{aligned}|a_h(w_h, v_h) - a(w_h, v_h)| &\leqslant c(v_1 - v_2) \sum_{T \in \tau_h} \|\nabla w_h\|_{L^2(\omega_T)} \|\nabla v_h\|_{L^2(\omega_T)} \\ &\leqslant C(v_1 - v_2) \sum_{T \in \tau_h} h\|\nabla w_h\|_{L^2(T)} \|\nabla v_h\|_{L^2(T)} \\ &\leqslant C(v_1 - v_2)h\|w_h\|_V \|v_h\|_V,\end{aligned}$$

which implies (15). □

Now we can state the

Theorem 3. *Let* $(u, p) \in V \times Q$ *and* $(u_h, p_h) \in V_h \times Q_h$ *be solutions of* (1) *and* (11), *respectively. Then, we have*

$$\|u - u_h\|_V + \|p - p_h\|_Q \leqslant M_0 h\|u\|_V + M_1 \inf_{v_h \in V_h} \|u - v_h\|_V + M_2 \inf_{q_h \in Q_h} \|p - q_h\|_Q, \tag{16}$$

where M_0, M_1 *and* M_2 *are positive constants given by*

$$\begin{aligned}M_0 &= \frac{c_1}{\delta_4} + \frac{c_1}{v_2\delta_2}\left(1 + \frac{c_1 h_0}{\delta_4} + \frac{v_1\delta_1}{\delta_4}\right), \\ M_1 &= \left(1 + \frac{\delta_3}{\delta_4}\right)\left(1 + \frac{c_1 h_0}{v_2\delta_2} + \frac{v_1\delta_1}{v_2\delta_2}\right)\left(1 + \frac{c_1 h_0}{\delta_4} + \frac{v_1\delta_1}{\delta_4}\right), \\ M_2 &= 1 + \frac{\delta_3}{\delta_4} + \frac{\delta_3}{v_2\delta_2}\left(1 + \frac{c_1 h_0}{\delta_4} + \frac{v_1\delta_1}{\delta_4}\right)\end{aligned}$$

with $c_1 = c_0(v_1 - v_2)$.

Proof. Let $(u, p) \in V \times Q$ and $(u_h, p_h) \in V_h \times Q_h$ be solutions of (1) and (11), respectively. Put $\delta_i' = v_i\delta_i$ for $i = 1, 2$. We note firstly that

$$\begin{cases} a(u, v_h) - a_h(u_h, v_h) + b(v_h, p - p_h) = 0 & (\forall v_h \in V_h), \\ b(u - u_h, q_h) = 0 & (\forall q_h \in Q_h) \end{cases} \tag{17}$$

by (1) and (11). We introduce

$$X_h = \{v_h \in V_h \mid b(v_h, q_h) = 0 \ (\forall q_h \in Q_h)\}$$

and then take $w_h \in X_h$ and $q_h \in Q_h$. By virtue of (2), (4), (13), (15) and (17), we can estimate as

$$\begin{aligned}
\delta_2' \|w_h - u_h\|_V &\leqslant \sup_{v_h \in X_h} \frac{a_h(w_h - u_h, v_h)}{\|v_h\|_V} \\
&\leqslant \sup_{v_h \in X_h} \frac{|a_h(w_h, v_h) - a(w_h, v_h)| + |a(u - w_h, v_h)| + |b(v_h, p - q_h)|}{\|v_h\|_V} \\
&\leqslant c_1 h \|w_h\|_V + \delta_1' \|u - w_h\|_V + \delta_3 \|p - q_h\|_Q \\
&\leqslant c_1 h \|u\|_V + (c_1 h + \delta_1') \|u - w_h\|_V + \delta_3 \|p - q_h\|_Q .
\end{aligned}$$

This, together with the triangle inequality, implies

$$\|u - u_h\|_V \leqslant \frac{c_1 h}{\delta_2'} \|u\|_V + \left(1 + \frac{c_1 h}{\delta_2'} + \frac{\delta_1'}{\delta_2'}\right) \|u - w_h\|_V + \frac{\delta_3}{\delta_2'} \|p - q_h\|_Q .$$

On the other hand, by (9), (15) and (17), we get

$$\begin{aligned}
\delta_4 \|q_h - p_h\|_Q &\leqslant \sup_{v_h \in V_h} \frac{b(v_h, q_h - p_h)}{\|v_h\|_V} \\
&\leqslant \sup_{v_h \in V_h} \frac{|a_h(u_h, v_h) - a(u_h, v_h)| + |a(u - u_h, v_h)| + |b(v_h, p - q_h)|}{\|v_h\|_V} \\
&\leqslant c_1 h \|u\|_V + (c_1 h + \delta_1') \|u - u_h\|_V + \delta_3 \|p - q_h\|_Q ,
\end{aligned}$$

and hence

$$\|p - p_h\|_Q \leqslant \frac{c_1 h}{\delta_4} \|u\|_V + \left(\frac{c_1 h}{\delta_4} + \frac{\delta_1'}{\delta_4}\right) \|u - u_h\|_V + \left(1 + \frac{\delta_3}{\delta_4}\right) \|p - q_h\|_Q .$$

Summing up those estimates and using the well-known inequality (cf. [11])

$$\inf_{w_h \in X_h} \|u - w_h\|_V \leqslant \left(1 + \frac{\delta_3}{\delta_4}\right) \inf_{v_h \in V_h} \|u - v_h\|_V ,$$

we obtain (16). □

Remark 4. Theorem 3 remains valid for an arbitrary pair of conforming finite element spaces $V_h \subset V$ and $Q_h \subset Q$ satisfying (9) and (15). However, in view of Lemma 1, we can obtain (15) for $V_h = \{v_h \in C(\overline{\Omega})^2 \cap V \mid v_h \in \mathscr{P}_n(T)\ (\forall T \in \mathscr{T}_h)\}$ with $n \geqslant 1$. Thus, Theorem 3 remains valid for all pair of conforming finite element spaces V_h and Q_h satisfying the uniform inf–sup condition.

Remark 5. From (16), we can observe that we only obtain the error estimate of the order h, even if higher order elements are employed. This is not surprising, because Σ is approximated by the piecewise linear curve. However, as will be investigated in the subsequent sections, such error estimate of the order h is not always guaranteed, even when the solution (u, p) is sufficiently regular.

Remark 6. Put $\alpha = v_1/v_2$ and fix v_2. Then, we have $M_0 = \mathrm{O}(\alpha^2)$, $M_1 = \mathrm{O}(\alpha^2)$ and $M_2 = \mathrm{O}(\alpha)$ as $\alpha \to \infty$.

5. Convergence rates

This section is devoted to a study on rates of convergence of our approximate solutions. To this end, we need to make some assumptions on the regularity of a solution (u, p) of (1) and on the geometry of the triangulation $\mathscr{T}_h$. Throughout this section, we let $k = 1, 2$.

Firstly, concerning the regularity of a solution (u, p) of (1), it is natural to expect that

$$u|_{\Omega_k} \in H^2(\Omega_k)^2, \quad p|_{\Omega_k} \in H^1(\Omega_k). \tag{18}$$

When Σ is a closed C^3 curve and Ω is a convex polygon, (18) actually holds true. On the other hand, we cannot expect (18), when $\partial\Omega \cap \overline{\Sigma} \neq \emptyset$ and the maximum interior angle of $\partial\Omega_1$, $\partial\Omega_2$ at R_1, R_2 is large enough. For more details of these facts, see Appendix A.

From this reason, instead of (18), we assume that there exists $\varepsilon \in (0, 1]$ such that

$$u|_{\Omega_k} \in H^{1+\varepsilon}(\Omega_k)^2, \quad p|_{\Omega_k} \in H^{\varepsilon}(\Omega_k). \tag{19}$$

We can find no explicit reference to derive any regularity results of the form (19). Nevertheless, we consider (19) on the analogy of elliptic interface problem; for more detail, see Appendix A.

We recall fundamental results for $H^s(\Omega)$ (cf. [15, Chapitre 1]): if $s \in (0, \frac{1}{2})$, then

$$\begin{aligned}
&\chi \in L^2(\Omega), \quad \chi|_{\Omega_k} \in H^s(\Omega_k) \quad \Leftrightarrow \quad \chi \in H^s(\Omega);\\
&\chi \in H^1(\Omega), \quad \chi|_{\Omega_k} \in H^{1+s}(\Omega_k) \quad \Leftrightarrow \quad \chi \in H^{1+s}(\Omega).
\end{aligned}$$

Hence, if (19) holds for some $\varepsilon \in (0, \frac{1}{2})$, we have the global regularity

$$u \in H^{1+\varepsilon}(\Omega)^2, \quad p \in H^{\varepsilon}(\Omega).$$

Then, we can apply the standard estimates (cf. [7, Theorem 2.27], [11, Lemma I.A.5] etc.)

$$\inf_{v_h \in V_h} \|u - v_h\|_V \leqslant Ch^{\varepsilon}\|u\|_{H^{1+\varepsilon}(\Omega)} \quad (u \in H^{1+\varepsilon}(\Omega)^2), \tag{20}$$

$$\inf_{q_h \in Q_h} \|p - q_h\|_Q \leqslant Ch^{\varepsilon}\|p\|_{H^{\varepsilon}(\Omega)} \quad (p \in H^{\varepsilon}(\Omega)), \tag{21}$$

and establish the following theorem.

Theorem 7. *In addition to the assumptions of Theorem* 3, *we assume* (19) *for some* $\varepsilon \in (0, \frac{1}{2})$. *Then we have*

$$\|u - u_h\|_V + \|p - p_h\|_Q \leqslant Ch^{\varepsilon}(\|u\|_{H^{1+\varepsilon}(\Omega)} + \|p\|_{H^{\varepsilon}(\Omega)}), \tag{22}$$

where $C = C(\varepsilon, \Omega_1, \Omega_2, h_0, M_0, M_1, M_2) > 0$ *is a constant. In particular, we can take as*

$$C = M_0 h_0^{1-\varepsilon} + C' \max\{M_1, M_2\}$$

with $C' = C'(\varepsilon, \Omega_1, \Omega_2) > 0$.

Remark 8. Theorem 7 remains true for an arbitrary pair of conforming finite element spaces V_h and Q_h satisfying (9), (15), (20) and (21). In view of Remark 4, all popular elements are applicable.

We proceed to the case $\varepsilon \in [\frac{1}{2}, 1]$. In this case, $u|_{\Omega_k} \in H^{1+\varepsilon}(\Omega_k)^2$ does not imply $u \in H^{1+\varepsilon}(\Omega)^2$. However, we can deduce explicit convergence rates with the aid of a geometrical condition on $\mathscr{T}_h$. Thus we introduce the condition that

$$P_T^1 \text{ and } P_T^2 \text{ coincide with some vertices of } T \text{ for any } T \in \tau_h. \tag{23}$$

This is equivalent to assume that l_T is coincident with an edge of T for any $T \in \tau_h$. Under such assumption, Ω is divided into two disjoint subdomains $\Omega_{k,h}$ defined as

$$\overline{\Omega_{k,h}} = \bigcup \{T \in \mathscr{T}_h \mid T \backslash \overline{\varpi_T} \subset \Omega_k \text{ if } T \in \tau_h,\ T \subset \overline{\Omega_k} \text{ if } T \notin \tau_h\}, \tag{24}$$

which are polygonal approximations of Ω_k, and

$$v_h(x) = v_k \quad (x \in \Omega_{k,h}).$$

See Fig. 3. Moreover every node belonging to $\partial\Omega_{1,h} \cap \partial\Omega_{2,h}$ is located on Σ. The same assumption is considered in, for example, [4,6,8,10,22].

Now, under those assumptions, we can establish the

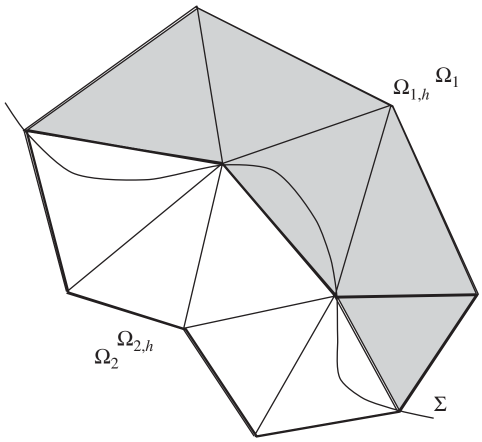

Fig. 3. $\Omega_{1,h}$ and $\Omega_{2,h}$ under the condition (23).

Theorem 9. *In addition to the assumptions of Theorem* 3, *we assume that* (19) *holds for some* $\varepsilon \in [\frac{1}{2}, 1]$, *and moreover suppose that* (23). *Then*

$$\|u-u_h\|_V + \|p-p_h\|_Q \leqslant Ch^{\varepsilon} \sum_{k=1}^{2} (\|u\|_{H^{1+\varepsilon}(\Omega_k)} + \|p\|_{H^{\varepsilon}(\Omega_k)}), \tag{25}$$

where $C = C(\varepsilon, \Omega_1, \Omega_2, h_0, M_0, M_1, M_2) > 0$. *In particular, we can take as* $C = M_0 h_0^{1-\varepsilon} + C' \max\{M_1, M_2\}$ *with* $C' = C'(\varepsilon, \Omega_1, \Omega_2) > 0$.

The proof is a direct consequence of Lemmas 10 and 11 described below.

Lemma 10 is concerned with the interpolation operator $J_h : C(\overline{\Omega})^2 \cap V \to V_h$ corresponding to $\mathscr{T}_h$ defined by

$$\begin{cases} J_h v(P) = v(P) & \text{for all nodes } P \text{ of } \mathscr{T}_h, \\ J_h v(G_T) = v(G_T) & \text{for the barycenter } G_T \text{ of } T \in \mathscr{T}_h. \end{cases}$$

We recall (cf. [7])

$$\|\varphi - J_h \varphi\|_{H^1(\Omega_{k,h})} \leqslant Ch^s \|\varphi\|_{H^{1+s}(\Omega_{k,h})} \quad (\varphi \in H^{1+s}(\Omega)^2,\ s \in (0,1]). \tag{26}$$

Lemma 10. *Let* $\varepsilon \in (0,1]$ *and suppose* (23) *holds. Then, for any* $u \in V$ *satisfying* $u|_{\Omega_k} \in H^{1+\varepsilon}(\Omega_k)^2$, *we have* $u \in C(\overline{\Omega})^2$ *and*

$$\|u - J_h u\|_V \leqslant Ch^{\varepsilon}(\|u\|_{H^{1+\varepsilon}(\Omega_1)} + \|u\|_{H^{1+\varepsilon}(\Omega_2)}) \tag{27}$$

with $C = (\varepsilon, \Omega_1, \Omega_2) > 0$.

Proof. The first assertion is obvious. To prove the second one, we follow the method of [22] and [23]. Set

$$\omega_{k,h} = \bigcup \{\omega_T \mid \omega_T \subset \Omega_k,\ T \in \tau_h\}. \tag{28}$$

Then, according to [22, Theorem 3.8], there exists $C = C(s, \Omega_k) > 0$ satisfying

$$\|\chi\|_{H^1(\omega_{k,h})} \leqslant Ch^s \|\chi\|_{H^{1+s}(\Omega_k)} \quad (\chi \in H^{1+s}(\Omega_k),\ s \in (0,1]). \tag{29}$$

In view of extension theorems and interpolation theorems (cf. [1]), we can take $\tilde{u}_k \in H^{1+\varepsilon}(\Omega)$ such that $\tilde{u}_k = u$ in Ω_k and

$$\|\tilde{u}_k\|_{H^{1+\varepsilon}(\Omega)} \leqslant C\|u\|_{H^{1+\varepsilon}(\Omega_k)} \tag{30}$$

with $C = C(\Omega_k) > 0$. Let $m = 1, 2$ such that $m \neq k$. Then, (23) assures that $u = \tilde{u}_k$ in $\Omega_{k,h} \backslash \omega_{m,k}$. Moreover, in view of (10), we may suppose that G_T (=the barycenter of T) $\notin \omega_{m,k}$ for any $T \subset \overline{\Omega_{k,h}}$. Therefore we have $J_h u = J_h \tilde{u}_k$ in $\Omega_{k,h}$. Hence, by (26), (29) and (30),

$$\begin{aligned}\|u - J_h u\|_{H^1(\Omega_{k,h})} &\leqslant \|u - \tilde{u}_k\|_{H^1(\Omega_{k,h})} + \|\tilde{u}_k - J_h u\|_{H^1(\Omega_{k,h})} \\ &= \|u - \tilde{u}_k\|_{H^1(\omega_{m,h})} + \|\tilde{u}_k - J_h \tilde{u}_k\|_{H^1(\Omega_{k,h})} \\ &\leqslant Ch^{\varepsilon}\|u - \tilde{u}_k\|_{H^{1+\varepsilon}(\Omega_m)} + Ch^{\varepsilon}\|\tilde{u}_k\|_{H^{1+\varepsilon}(\Omega_{k,h})} \\ &\leqslant Ch^{\varepsilon}(\|u\|_{H^{1+\varepsilon}(\Omega_1)} + \|u\|_{H^{1+\varepsilon}(\Omega_2)}),\end{aligned}$$

which implies (27). □

We proceed to the estimation of the pressure part. To this end, we introduce the local L^2 projection operator $\pi_h : Q \to Q_h$ defined as

$$\pi_h \chi|_T \in \mathscr{P}_1(T), \quad \int_T (\pi_h \chi - \chi)\psi \, dx = 0 \quad (\forall \psi \in \mathscr{P}_1(T), \ \forall T \in \mathscr{T}_h).$$

We recall (cf. [11]) that there is $C = C(s, \Omega) > 0$ such that

$$\|\chi - \pi_h \chi\|_{L^2(\Omega)} \leqslant Ch^s \|\chi\|_{H^s(\Omega)} \quad (\chi \in H^s(\Omega), \ s \in [0, 2]).$$

Lemma 11. *Let* $\varepsilon \in (0, 1]$ *and suppose that* (23). *Then, there exists* $C = C(\varepsilon, \Omega_1, \Omega_2) > 0$ *such that*

$$\|p - \pi_h p\|_Q \leqslant Ch^{\varepsilon}(\|p\|_{H^{\varepsilon}(\Omega_1)} + \|p\|_{H^{\varepsilon}(\Omega_2)}) \tag{31}$$

for any $p \in Q$ *satisfying* $p|_{\Omega_k} \in H^{\varepsilon}(\Omega_k)$.

Proof. We again use $\Omega_{k,h}$ and $\omega_{k,h}$ defined by (24) and (28). Let $m = 1, 2$ with $m \neq k$. In this case, instead of (29), we apply

$$\|\chi\|_{L^2(\omega_{k,h})} \leqslant Ch^s \|\chi\|_{H^s(\Omega_k)} \quad (\chi \in H^s(\Omega_k), \ s \in (0, 1]) \tag{32}$$

with $C = C(s, \Omega_k) > 0$. (Although the proof of (32) is not explicitly stated in [22], it could be done in the same way as that of (29) with obvious modifications.) We note, by the definition, that

$$\|\pi_h \chi\|_{L^2(T)} \leqslant \|\chi\|_{L^2(T)} \quad (\chi \in L^2(\Omega), \ T \in \mathscr{T}_h).$$

Now let $\tilde{p}_k \in H^{\varepsilon}(\Omega)$ be an extension of $p|_{\Omega_k} \in H^{\varepsilon}(\Omega_k)$ subject to $\|\tilde{p}_k\|_{H^{\varepsilon}(\Omega)} \leqslant C\|p\|_{H^{\varepsilon}(\Omega_k)}$. Then, by (23), $p = \tilde{p}$ in $\Omega_{k,h} \backslash \omega_{m,h}$. Hence,

$$\begin{aligned}\|p - \pi_h p\|_{L^2(\Omega_{k,h})} &\leqslant \|p - \tilde{p}_k\|_{L^2(\Omega_{k,h})} + \|\tilde{p}_k - \pi_h \tilde{p}_k\|_{L^2(\Omega_{k,h})} + \|\pi_h(\tilde{p}_k - p)\|_{L^2(\Omega_{k,h})} \\ &\leqslant 2\|p - \tilde{p}_k\|_{L^2(\omega_{m,h})} + \|\tilde{p}_k - \pi_h \tilde{p}_k\|_{L^2(\Omega_{k,h})} \\ &\leqslant Ch^{\varepsilon}\|p - \tilde{p}_k\|_{H^{\varepsilon}(\Omega_m)} + Ch^{\varepsilon}\|\tilde{p}_k\|_{H^{\varepsilon}(\Omega_{h,k})} \\ &\leqslant Ch^{\varepsilon}(\|p\|_{H^{\varepsilon}(\Omega_1)} + \|p\|_{H^{\varepsilon}(\Omega_2)}),\end{aligned}$$

which completes the proof. □

6. Extension to P1-iso-P2/P1 element

We introduce another triangulation $\mathscr{S}_h$ by dividing each triangle $T \in \mathscr{T}_h$ into four equal triangles by segments connecting midpoints of the edge of T. We take

$$\tilde{V}_h = \{v_h \in C(\overline{\Omega})^2 \cap V \mid v_h|_K \in \mathscr{P}_1(K)^2 \ (\forall K \in \mathscr{S}_h)\}.$$

It is well-known that a pair of $\tilde{V}_h$ and Q_h satisfies the uniform inf–sup condition (9). Let

$$\sigma_h = \{K \in \mathscr{S}_h |\ (\mathrm{Int}\, K) \cap \Sigma \neq \emptyset\}.$$

Again, by re-choosing h_0 if necessary, we may suppose that ∂K and Σ connect by only two points P_K^1 and P_K^2 when $h \in (0, h_0)$. For each $K \in \sigma_h$, we define $l_K, \omega_K, K_1, K_2, K_1^h$ and K_2^h in the similar way as before. Then $\bigcup_{K\in\sigma_h} l_K$ is another piecewise linear approximation of Σ.

In this case, we consider the problem to find $(u_h, p_h) \in \tilde{V}_h \times Q_h$ satisfying

$$\begin{cases} \tilde{a}_h(u_h, v_h) + b(v_h, p_h) = (f, v_h) & (\forall v_h \in \tilde{V}_h), \\ b(u_h, q_h) = 0 & (\forall q_h \in Q_h), \end{cases} \tag{33}$$

where

$$\tilde{a}_h(u_h, v_h) = \int_\Omega 2\tilde{v}_h(x) D_{ij}(u_h) D_{ij}(v_h)\, \mathrm{d}x \quad (u_h, v_h \in \tilde{V}_h)$$

and

$$\tilde{v}_h(x) = \begin{cases} v(x) & (x \in \mathrm{Int}\, K,\ \ K \in \mathscr{S}_h \backslash \sigma_h), \\ v_k & (x \in \mathrm{Int}\, K_k^h,\ \ k = 1, 2,\ \ K \in \sigma_h). \end{cases}$$

Since functions of $\tilde{V}_h$ are piecewise linear, a more convenient choice of approximation of $v(x)$ is possible. That is, introducing

$$\hat{v}_h(x) = \begin{cases} v(x) & (x \in \mathrm{Int}\, K,\ \ K \in \mathscr{S}_h \backslash \sigma_h), \\ (1-\theta_K)v_1 + \theta_K v_2 & (x \in \mathrm{Int}\, K,\ \ K \in \sigma_h), \end{cases} \tag{34}$$

where $\theta_K =$ (the area of K_2^h)/(the area of K), we put

$$\hat{a}_h(u_h, v_h) = \int_\Omega 2\hat{v}_h(x) D_{ij}(u_h) D_{ij}(v_h)\, \mathrm{d}x \quad (u_h, v_h \in \tilde{V}_h).$$

Then, we have, for all $u_h, v_h \in \tilde{V}_h$,

$$\tilde{a}_h(u_h, v_h) - \hat{a}_h(u_h, v_h) = \sum_{K\in\sigma_h} 2D_{ij}(u_h) D_{ij}(v_h) \int_K (v_h(x) - \hat{v}_h(x))\, \mathrm{d}x = 0.$$

Therefore (33) is equivalent to the following problem: find $(u_h, p_h) \in \tilde{V}_h \times Q_h$ satisfying

$$\begin{cases} \hat{a}_h(u_h, v_h) + b(v_h, p_h) = (f, v_h) & (\forall v_h \in \tilde{V}_h), \\ b(u_h, q_h) = 0 & (\forall q_h \in Q_h). \end{cases}$$

According to Remark 8, Theorems 3 and 7 remain true in this case. Moreover we have the

Theorem 12. *Let* $(u, p) \in V \times Q$ *and* $(u_h, p_h) \in \tilde{V}_h \times Q_h$ *be solutions of* (1) *and* (33), *respectively. If* (19) *holds for some* $\varepsilon \in [\frac{1}{2}, 1]$ *and* (23) *holds, we then obtain* (25).

Proof. It suffices to show that there exists $v_h \in \tilde{V}_h$ satisfying

$$\|u - v_h\|_V \leqslant C h^\varepsilon (\|u\|_{H^{1+\varepsilon}(\Omega_1)} + \|u\|_{H^{1+\varepsilon}(\Omega_2)}). \tag{35}$$

We introduce the linear interpolation operator $I_h : C(\overline{\Omega})^2 \cap V \to [Q_h \cap V]^2$ corresponding to $\mathscr{T}_h$ defined by

$$I_h v(P) = v(P) \quad \text{for all nodes } P \text{ of } \mathscr{T}_h,$$

Then $I_h u \in \tilde{V}_h$ and we have (26) for I_h. Hence the same argument as in the proof of Lemma 10 works. Thus we obtain (35) for $v_h = I_h u$. $\square$

Remark 13. In actual computations, $v(x)$ is often approximated by a constant on each element. Thus, Theorem 12 gives a certain justification of such approximation.

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Appendix A. Review on regularity results for a Stokes interface problem

In this appendix, we briefly review regularity results for the solution $(u, p) \in V \times Q$ of (1). Let $x_0 \in \overline{\Omega}$, and let $\mathcal{O}$ be a neighbourhood of x_0. Further let $k = 1, 2$. We then set $U = \mathcal{O} \cap \Omega$ and $U_k = U \cap \Omega_k$. We recall that R_1 and R_2 are the intersection points of $\partial\Omega$ and $\overline{\Sigma}$. It suffices to consider the following five cases:

(i) $x_0 \in \Omega_k$ and $\overline{U} \subset \Omega_k$;
(ii) $x_0 \in \partial\Omega$, x_0 is not a corner of $\partial\Omega$, $U \subset \Omega_k$, and $\overline{U}$ contains no other corners of $\partial\Omega$;
(iii) $x_0 \in \partial\Omega$, x_0 is a corner of $\partial\Omega$, $U \subset \Omega_k$, and $\overline{U}$ contains no other corners of $\partial\Omega$;
(iv) $x_0 \in \Sigma$, $\overline{U} \subset \Omega$, and $R_1, R_2 \notin U$;
(v) $x_0 = R_1, R_2$, and U contains no corners of $\partial\Omega$.

In the case of (i) and (ii), it is well-known that

$$u \in H^2(U)^2, \quad p \in H^1(U). \tag{A.1}$$

In the case of (iii), if the interior angle θ of $\partial\Omega$ at x_0 is less than π, then we have (A.1). See, for example, [14]. When θ is greater than π, combining [3, Théorème II.1] with [12, Theorem 1.2.18], we have a constant $\alpha = \alpha(\theta) \in (0, 1)$ such that

$$u \in H^{1+\alpha}(U)^2, \quad p \in H^{\alpha}(U).$$

In the case of (iv), as is described in [20], we can deduce

$$u \in H^2(U_k)^2, \quad p \in H^1(U_k). \tag{A.2}$$

(It can be verified by the standard regularization argument using difference quotients.)

However, in the case of (v), we have no information at present.

As an illustration of the regularity near intersection points, we recall a regularity result for a Poisson interface problem due to Petzoldt [18]. Let $w \in H_0^1(\Omega)$ be the unique solution of

$$\int_\Omega \nu(x)\nabla w \cdot \nabla v \, dx = \int_\Omega g v \, dx \quad (\forall v \in H_0^1(\Omega)),$$

where $g \in L^2(\Omega)$. We suppose that (v) holds, and let θ be the maximum interior angle of $\partial\Omega_1$ and $\partial\Omega_2$ at x_0. Then we have (cf. [18, Theorem 6.2])

$$w \in H^{1+\alpha}(U_k), \quad \alpha = \min\left\{1, \frac{\pi}{2\theta}\right\} \in (0, 1]. \tag{A.3}$$

Remark 14. (A.3) indicates that (A.2) can be expected only when θ is small enough. In fact, if (A.2) is true for a large θ, it contradicts (A.3).

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